Pareto Optimality Between Prescribed Far-Field Pattern and Radiation Efficiency

(Very) Preliminary Results

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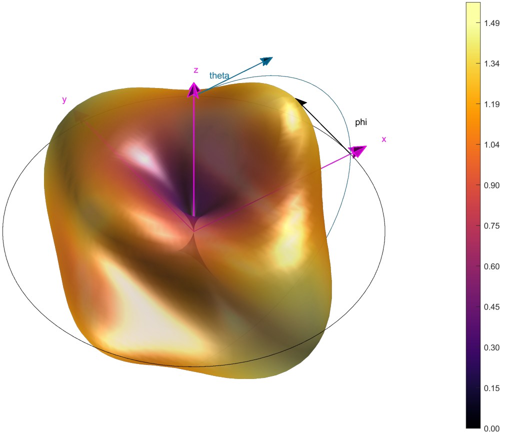
Workshop
Les Marécottes

Outline



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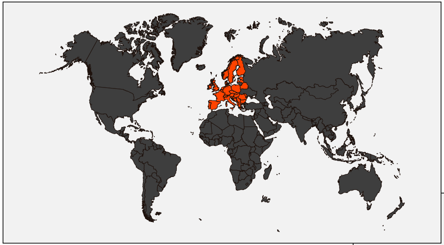

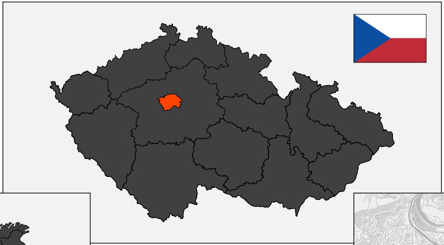

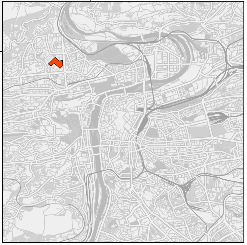

Czech Technical University in Prague

Established in 1707 as the first non-military technical university in Europe.

- From 12 students in 1707 to more than 20 000 students around 2020.

Left: Prague; right: CTU, Faculty of Electrical Engineering (one of eight faculties).

You are welcome to visit us in Prague!

Current Density Bounds

- Draw whatever current you want to extremize a given metric $f(\mathbf{I})$.

$$\begin{aligned} \underset{\mathbf{I}}{\text{minimize}} \quad & f(\mathbf{I}) \\ \text{subject to} \quad & g_i(\mathbf{I}) \leq c_i \end{aligned}$$

- Typically QCQP (or SDP).
- Full quadratic forms ...
- Substructures, port modes, ...

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Current Density Bounds

- Advanced for many scalar metrics, *e.g.*,
  - Q-factor[1] (bandwidth),
  - gain[2],
  - scattering[3],
  - optics[4],
  - realized gain[5],
  - trade-offs[6],
  - ...

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[1] M. Capek, M. Gustafsson, and K. Schab, "Minimization of antenna quality factor," *IEEE Trans. Antennas Propag.*, vol. 65, no. 8, pp. 4115–4123, 2017. DOI: 10.1109/TAP.2017.2717478

[2] M. Gustafsson and M. Capek, "Maximum gain, effective area, and directivity," *IEEE Trans. Antennas Propag.*, vol. 67, no. 8, pp. 5282 –5293, 2019. DOI: 10.1109/TAP.2019.2916760

[3] M. Gustafsson, K. Schab, L. Jelinek, *et al.*, "Upper bounds on absorption and scattering," *New Journal of Physics*, vol. 22, no. 7, p. 073 013, 2020. DOI: 10.1088/1367-2630/ab83d3

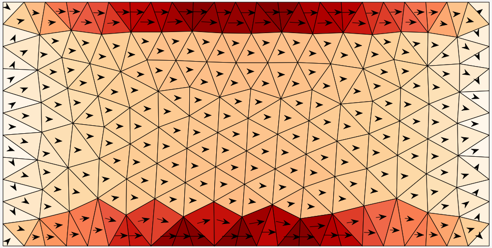
[4] K. Schab, L. Jelinek, M. Capek, *et al.*, "Upper bounds on focusing efficiency," *Optics Express*, vol. 30, no. 25, p. 45 705, Dec. 2022

[5] M. Capek, L. Jelinek, and M. Masek, "Finding optimal total active reflection coefficient and realized gain for multi-port lossy antennas," *IEEE Transactions on Antennas and Propagation*, vol. 69, no. 5, pp. 2481–2493, 2021. DOI: 10.1109/TAP.2020.3030941

[6] K. Schab, A. Rothschild, K. Nguyen, *et al.*, "Trade-offs in absorption and scattering by nanophotonic structures," *Optics Express*, vol. 28, pp. 36 584–36 599, 24 2020. DOI: 10.1364/OE.410520

A Note: MoM Solution × Current Impressed in Vacuum

MoM solution



Solution to $\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V}$ for an incident plane wave.

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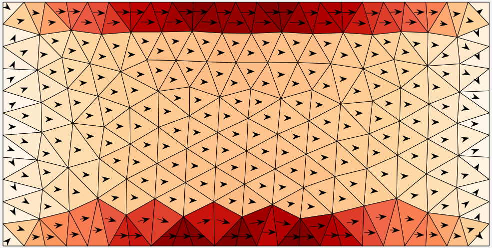

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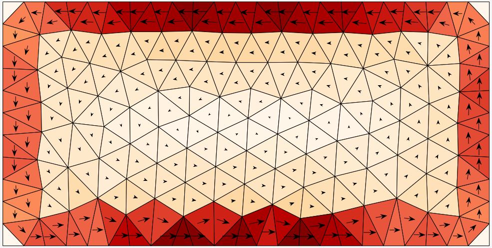

Solution to $\mathbf{X}\mathbf{I}_i = \lambda_i \mathbf{R}\mathbf{I}_i$ (the first inductive mode).

- Looking for an optimal current, it can be chosen completely freely, only the excitation $\mathbf{V} = \mathbf{Z}\mathbf{I}$ may not be realizable.

Far-field Optimality

How to deal with far-field optimality?

- Point-wise, *i.e.*, directivity $D(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}})$ or gain $G(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}})$.
- Prescribed far-field $\boldsymbol{F} = \boldsymbol{F}(\vartheta, \varphi)$:
  - is a vector function,
  - with a (unknown) phase,
  - required smoothness $(\boldsymbol{F}(\vartheta, \varphi) \longrightarrow \boldsymbol{F}(\vartheta_p, \varphi_p))$.

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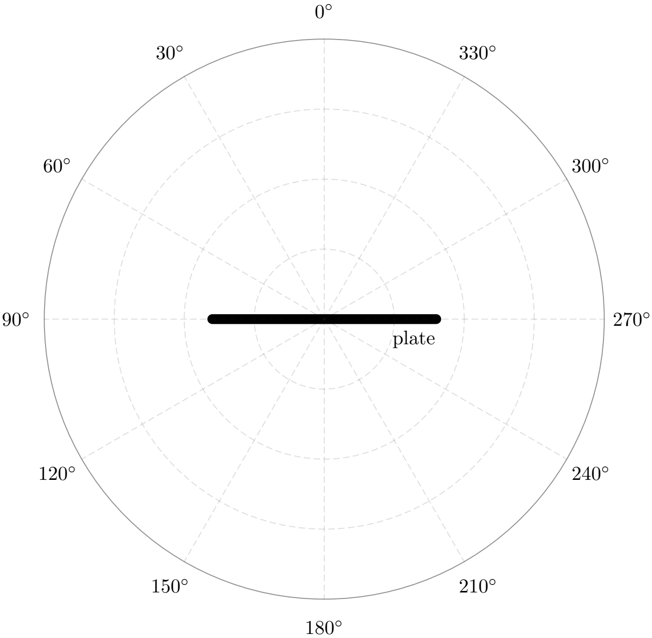
Some works exist

- For example, for the cost in Q-factor[7],
- far-field shaping for small antennas (SDP)[8],
- …

[7] M. Gustafsson and S. Nordebo, "Optimal antenna currents for Q, superdirectivity, and radiation patterns using convex optimization," *IEEE Trans. Antennas Propag.*, vol. 61, no. 3, pp. 1109–1118, 2013. DOI: 10.1109/TAP.2012.2227656

[8] S. Shi, L. Wang, and B. L. G. Jonsson, *Antenna current optimization and realizations for far-field pattern shaping*, 2017. [Online]. Available: https://arxiv.org/abs/1711.09709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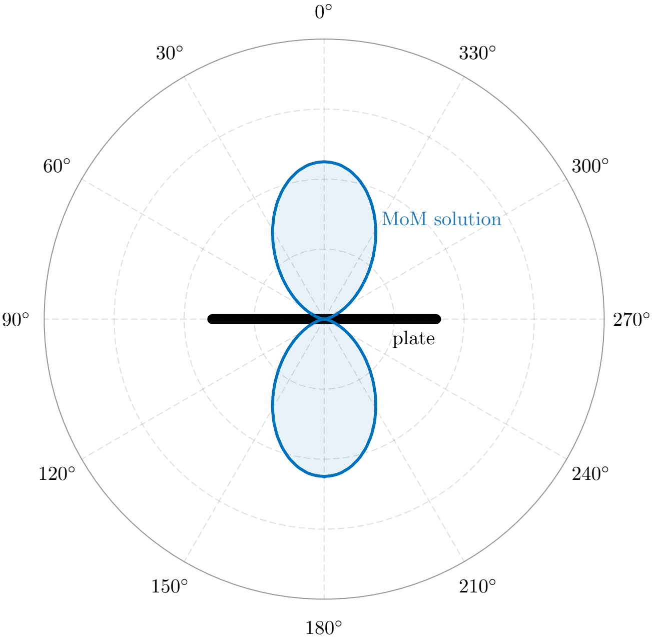
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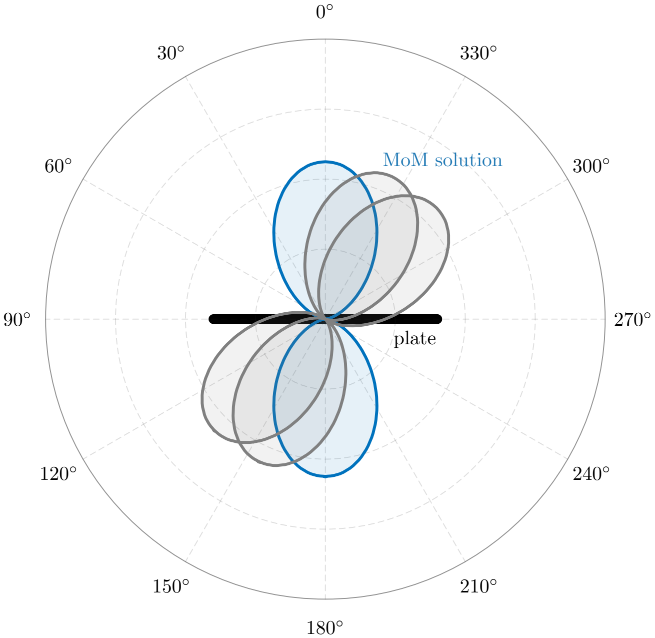
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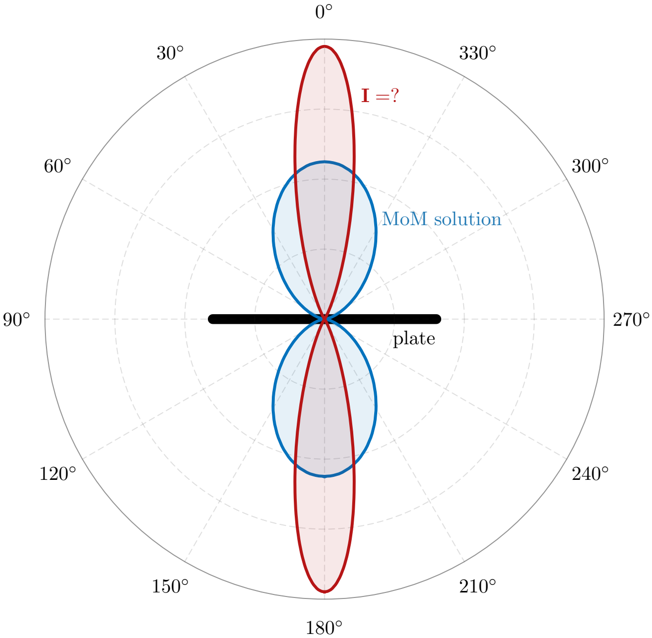
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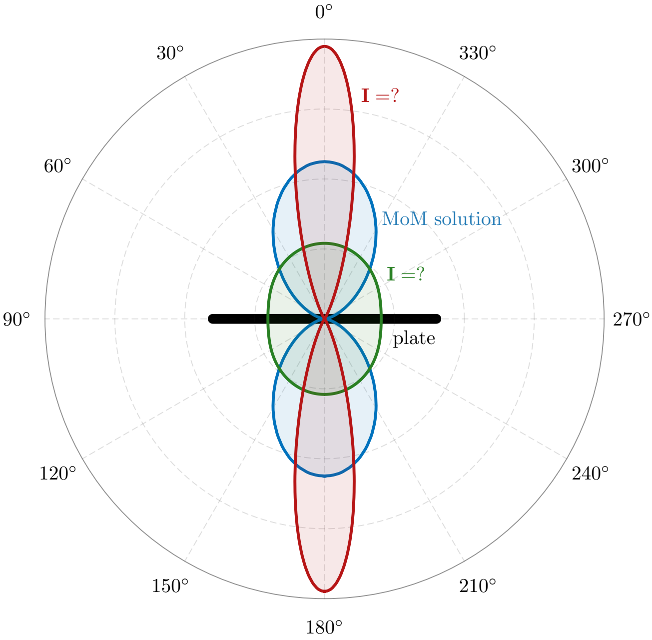
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Far-field Optimality as Trade-off With Radiation Efficiency

The hypothesis

"Almost every far field pattern F_0 can be generated by a current $\mathbf{I}_0$, however, potentially at the cost of almost zero radiation efficiency."

$$\begin{aligned} \underset{\mathbf{I}}{\text{minimize}} \quad & \varepsilon_{\mathrm{F}} = \left\| \boldsymbol{F}_0 - \boldsymbol{F}(\mathbf{I}) \right\| \\ \text{subject to} \quad & \eta_{\mathrm{rad}}(\mathbf{I}) \leq x \end{aligned}$$

- The problem above forms a Pareto frontier in $\varepsilon_{\mathrm{F}}(\mathbf{I})$ and $\eta_{\mathrm{rad}}(\mathbf{I})$.
- A type of norm taken $|\cdot|$ is crucial.

Far-field Optimality: A Role of Phase

Far field F_0 has to be given to solve

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Two posibilities differing in our knowledge of the desired far field $\boldsymbol{F}_0$:

Problem #1:

- Both amplitude and phase of $\boldsymbol{F}_0$:

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- Phase often arbitrary.

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Problem #2:

- Amplitude of $\boldsymbol{F}_0$ is known; phase is arbitrary:

$$\begin{aligned} \underset{\mathbf{I}}{\text{minimize}} \quad & \left\| |\boldsymbol{F}_0| - |\boldsymbol{F}(\mathbf{I})| \right\|^2 \\ \text{subject to} \quad & \eta_{\mathrm{rad}}\left(\mathbf{I}\right) \leq x \end{aligned}$$

- Hard to solve ($\propto$ MAX-CUT $\rightarrow$ NP-hard).

Methodology – Operators

- RWG representation of MoM IE operators.

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$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \begin{bmatrix} F\left(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{r}}\right) \\ F\left(\hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{r}}\right) \end{bmatrix}.$$

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- Far field component $F\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) = \mathbf{K}\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right)\mathbf{I}$, with $\mathbf{K} = \left[K_p\right]$ point-wise given as
$$K_p\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) = -\mathrm{j}\frac{Z_0 k}{4\pi}\int\limits_{\mathbb{R}^3} \hat{\boldsymbol{e}} \cdot \boldsymbol{\psi}_p\left(\boldsymbol{r}_1\right)\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}_1}\,\mathrm{d}V_1.$$

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- Impedance matrix

$$\mathbf{Z} = \mathbf{R} + \mathbf{L} + \mathrm{j}\mathbf{X}$$

with $\mathbf{R} + \mathrm{j}\mathbf{X}$ being a vacuum part, and $\mathbf{L}$ representing ohmic losses, point-wise as

$$L_{pq} = \int\limits_{\Omega} R_{\mathrm{s}}\left(\boldsymbol{r}\right)\boldsymbol{\psi}_p^*\left(\boldsymbol{r}\right) \cdot \boldsymbol{\psi}_q\left(\boldsymbol{r}\right)\,\mathrm{d}\Omega$$

(*e.g.*, thin-sheet model).

Methodology – Antenna Metrics

- Radiation efficiency

$$\eta_{\mathrm{rad}} = \frac{P_{\mathrm{rad}}}{P_{\mathrm{rad}} + P_{\mathrm{lost}}} \approx \frac{\mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I}}{\mathbf{I}^{\mathrm{H}} \left(\mathbf{R} + \mathbf{L}\right) \mathbf{I}} = \frac{1}{1+\delta} \tag{1}$$

  with $\delta = P_{\mathrm{lost}} / P_{\mathrm{rad}}$ being dissipation factor.

- Far field

$$F\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) \approx \mathbf{K}\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) \mathbf{I}.$$

- Antenna gain

$$G\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) = \frac{2\pi}{Z_0} \frac{\left|F\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right)\right|^2}{P_{\mathrm{rad}} + P_{\mathrm{lost}}} \approx \frac{4\pi}{Z_0} \frac{\mathbf{I}^{\mathrm{H}} \mathbf{K}^{\mathrm{H}}\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) \mathbf{K}\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) \mathbf{I}}{\mathbf{I}^{\mathrm{H}} \left(\mathbf{R} + \mathbf{L}\right) \mathbf{I}}.$$

Methodology – Far-Field Integration

- Radiation power

$$P_{\mathrm{rad}} = \frac{1}{2Z_0} \int\limits_{4\pi} \boldsymbol{F}^{*}\left(\hat{\boldsymbol{r}}\right) \cdot \boldsymbol{F}\left(\hat{\boldsymbol{r}}\right) \, \mathrm{d}\Omega \approx \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I}.$$

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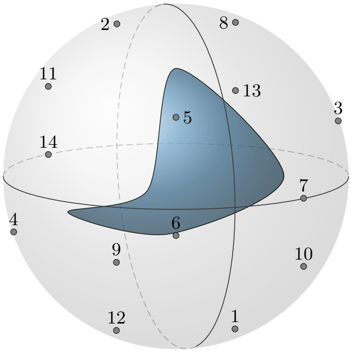
- Lebedev quadrature over unit ball

$$I = \int f(\Omega)\,\mathrm{d}\Omega \approx \sum_{n} \Lambda_n f(\vartheta_n, \varphi_n)$$

$$P_{\mathrm{rad}} \approx \frac{1}{2Z_0} \mathbf{I}^{\mathrm{H}} [\mathbf{K}]^{\mathrm{H}} \mathbf{\Lambda} [\mathbf{K}] \mathbf{I}.$$

with

$$[\mathbf{K}] = \begin{bmatrix} \mathbf{K}^{\mathrm{T}}(\vartheta_1, \varphi_1) & \cdots & \mathbf{K}^{\mathrm{T}}(\vartheta_n, \varphi_N) \end{bmatrix}^{\mathrm{T}}.$$



Lebedev quadrature with 14 points integrating exactly up to L_{max} (TM_{1m} and TE_{1m}).

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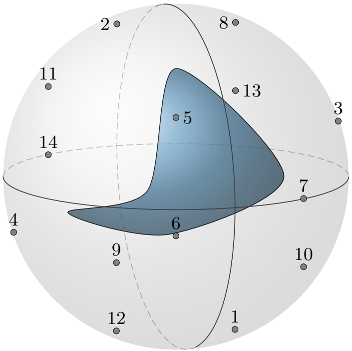

Lebedev quadrature with 14 points integrating exactly up to L_{max} (TM_{1m} and TE_{1m}).

- Analogy to guassian quadrature on spherical shell.
- Selected quadrature degree treats spherical harmonics exactly up to known order.

Problem #1 in RWG Basis

Let us focus on the **Problem #1** first. (Phase of $\boldsymbol{F}_0$ is specified).

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\begin{aligned}
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\text{subject to} \quad & \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{L} \mathbf{I} = \delta \\
& \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I} = 1
\end{aligned}
$$

- Quadratic program with two quadratic constraints.
- Relatively complicated optimized metric.
- The problem can rewritten preserving its original nature...

Problem #1 in RWG Basis – Simplification (Part 1)

Optimized metric is to be simplified

$$\frac{1}{2Z_0}\left(\mathbf{F}_0^{\mathrm{H}} - \mathbf{I}^{\mathrm{H}}[\mathbf{K}]^{\mathrm{H}}\right)\boldsymbol{\Lambda}\left(\mathbf{F}_0 - [\mathbf{K}]\mathbf{I}\right) = \frac{1}{2Z_0}\mathbf{F}_0^{\mathrm{H}}\boldsymbol{\Lambda}\mathbf{F}_0 - \frac{1}{Z_0}\mathrm{Re}\left(\mathbf{I}^{\mathrm{H}}[\mathbf{K}]^{\mathrm{H}}\boldsymbol{\Lambda}\mathbf{F}_0\right) + \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I}.$$

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Normalization of far field $\boldsymbol{F}_0$

Let us assume for the rest of the talk that the desired far field is normalized so that

$$P_{\mathrm{rad},0} \approx \frac{1}{2Z_0}\mathbf{F}_0^{\mathrm{H}}\mathbf{\Lambda}\mathbf{F}_0 = 1$$

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Envelope correlation coefficient

$$E(\boldsymbol{F}, \boldsymbol{F}_0) = |\rho(\boldsymbol{F}, \boldsymbol{F}_0)|^2 = \left|\frac{\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I}_0}{\sqrt{\mathbf{I}_0^{\mathrm{H}}\mathbf{R}\mathbf{I}_0\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I}}}\right|^2 = \cdots = |\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I}_0|^2$$

Problem #1 in RWG Basis – Simplification (Part 2)

$$\begin{aligned} \underset{\mathbf{I}}{\text{minimize}} \quad & 2 - \frac{1}{Z_0} \mathrm{Re}\left(\mathbf{I}^{\mathrm{H}} [\mathbf{K}]^{\mathrm{H}} \mathbf{\Lambda} \mathbf{F}_0\right) \\ \text{subject to} \quad & \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{L} \mathbf{I} = \delta \\ & \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I} = 1 \end{aligned}$$

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\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\underset{\mathbf{I}}{\text{maximize}} \quad & \frac{1}{Z_0} \mathrm{Re}\left(\mathbf{I}^{\mathrm{H}} [\mathbf{K}]^{\mathrm{H}} \mathbf{\Lambda} \mathbf{F}_0\right) = \mathrm{Re}(\rho(\boldsymbol{F}, \boldsymbol{F}_0)) \\
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$$
\begin{aligned}
\underset{\mathbf{I}}{\text{minimize}} \quad & 2 - \frac{1}{Z_0} \mathrm{Re}\left(\mathbf{I}^{\mathrm{H}} [\mathbf{K}]^{\mathrm{H}} \mathbf{\Lambda} \mathbf{F}_0\right) \\
\text{subject to} \quad & \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{L} \mathbf{I} = \delta \\
& \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I} = 1
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\underset{\mathbf{I}}{\text{maximize}} \quad & \frac{1}{Z_0} \mathrm{Re}\left(\mathbf{I}^{\mathrm{H}} [\mathbf{K}]^{\mathrm{H}} \mathbf{\Lambda} \mathbf{F}_0\right) = \mathrm{Re}(\rho(\boldsymbol{F}, \boldsymbol{F}_0)) \\
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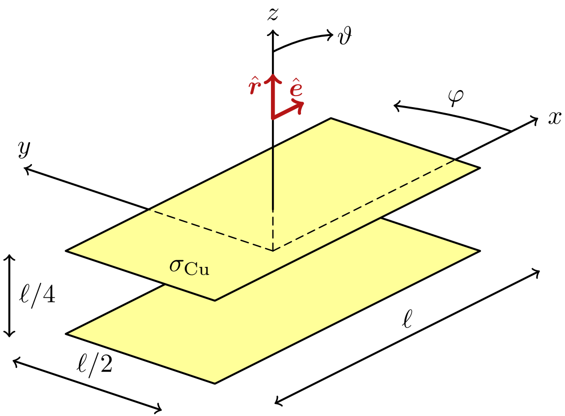
& \frac{1}{2} \mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I} = 1
\end{aligned}
$$

- In MIMO, ECC is usually minimized, here we want to maximize!
- A possibility to reduce to QCQP with one quadratic constraint only…
  - Grouping **R** and **L** constraints and changing multipliers.

Antenna Design Region

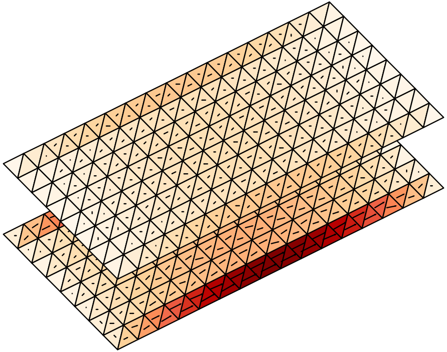


Two parallel plates, $k\ell = \pi$, copper $\sigma = 5.96 \cdot 10^7\,\mathrm{Sm}^{-1}$.

- Two plates shown above are used everywhere in this talk as an example.

Example #1: Synthesis of MoM Current

- Current $\mathbf{I}_0$ is evaluated for an impinging plane wave (normal incidence, $\hat{\boldsymbol{x}}$ polarization).



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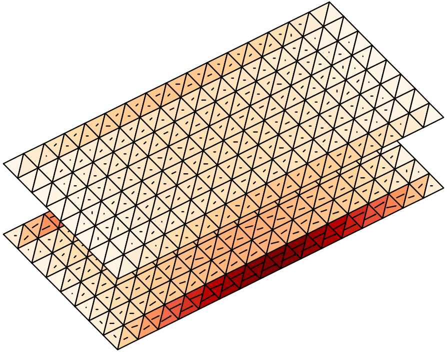

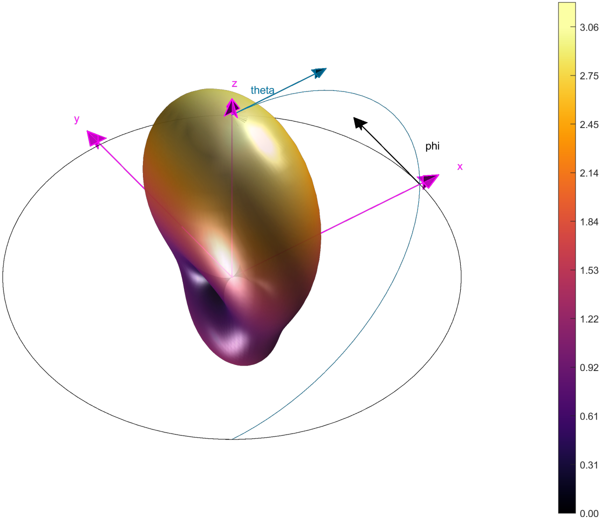

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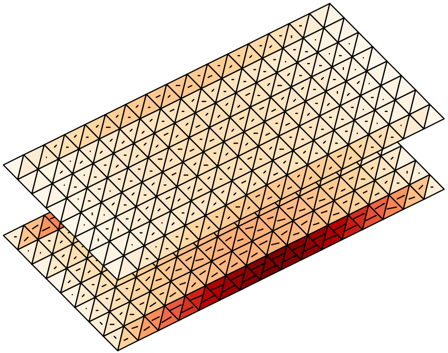
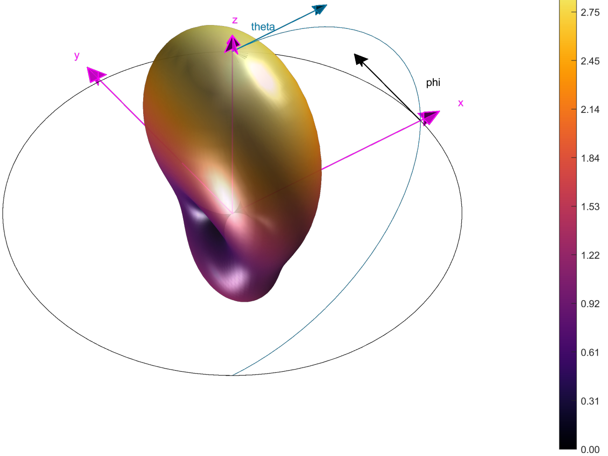
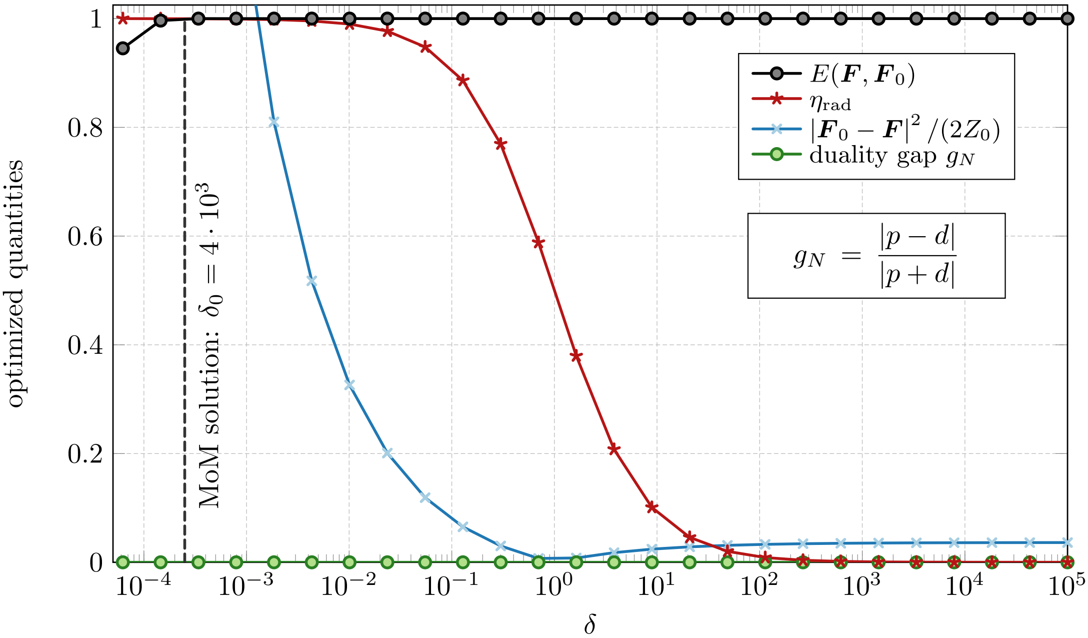
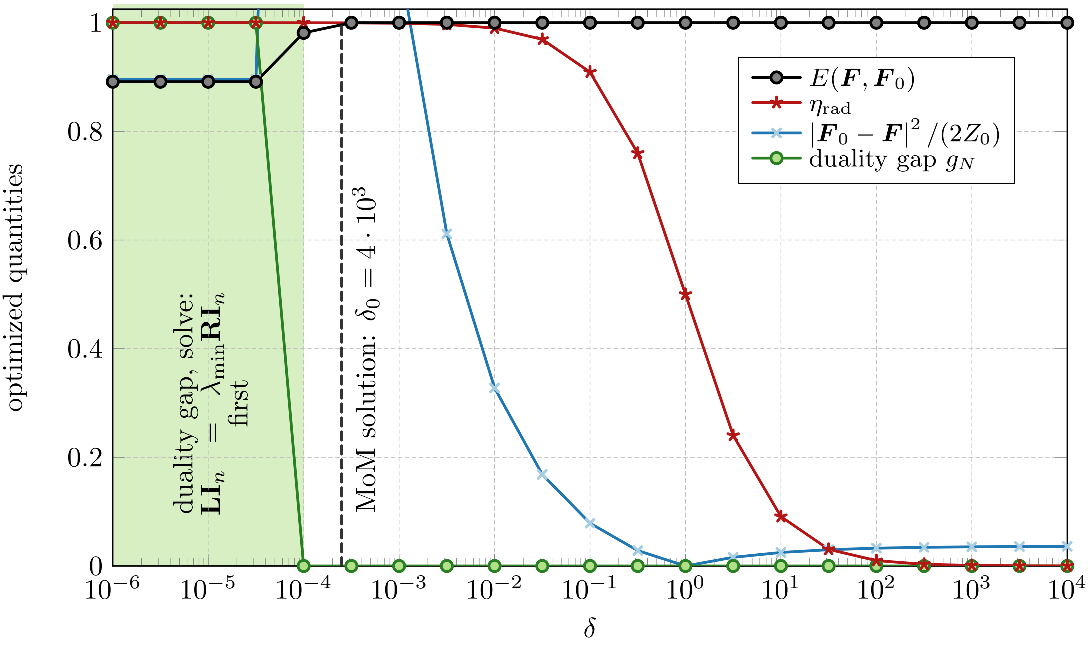





optimized quantities
δ
MoM solution: $\delta_0 = 4 \cdot 10^3$
$E(\boldsymbol{F}, \boldsymbol{F}_0)$
η_{rad}
$|\boldsymbol{F}_0 - \boldsymbol{F}|^2 / (2Z_0)$
duality gap g_N
$g_N = \frac{|p-d|}{|p+d|}$
0
0.2
0.4
0.6
0.8
1
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}



1
0.8
0.6
0.4
0.2
0
optimized quantities
10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4}
δ
duality gap, solve: $\mathbf{LI}_n = \lambda_{\min}\mathbf{RI}_n$ first
MoM solution: $\delta_0 = 4 \cdot 10^3$
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η_{rad}
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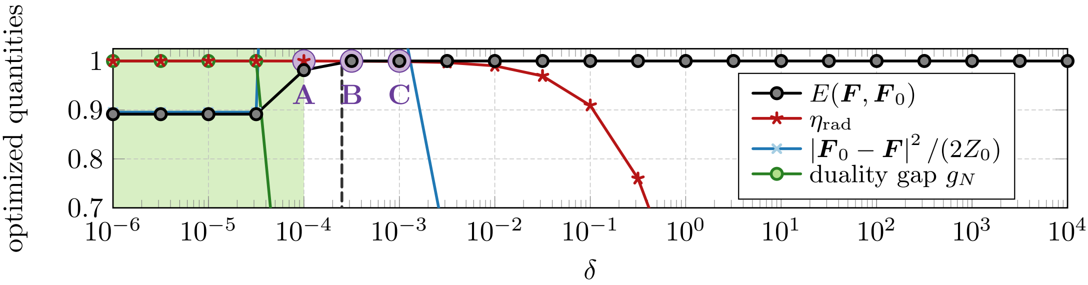
optimized quantities
1
0.9
0.8
0.7
10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4}
δ
A
B
C
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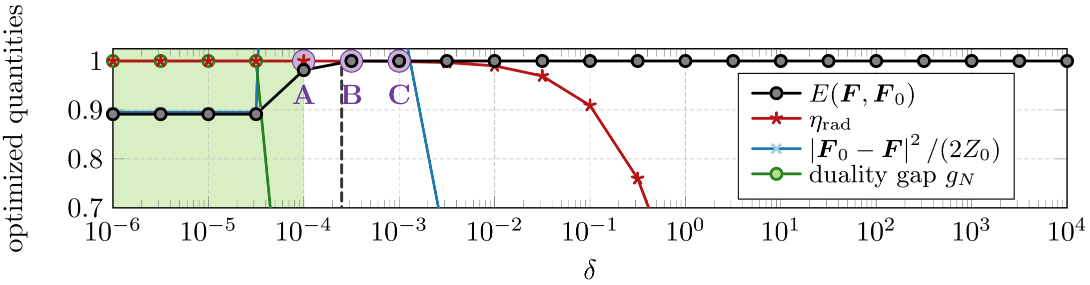
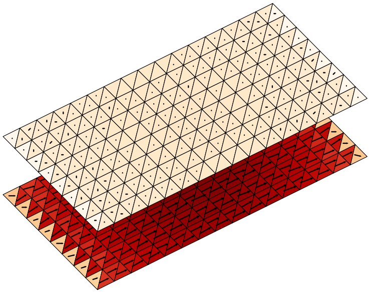
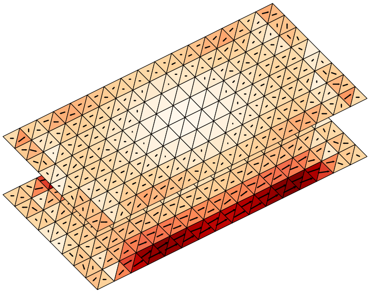
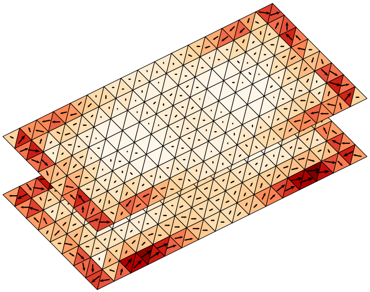




Optimal current for point **A** ($\delta \approx 10^{-4}$).

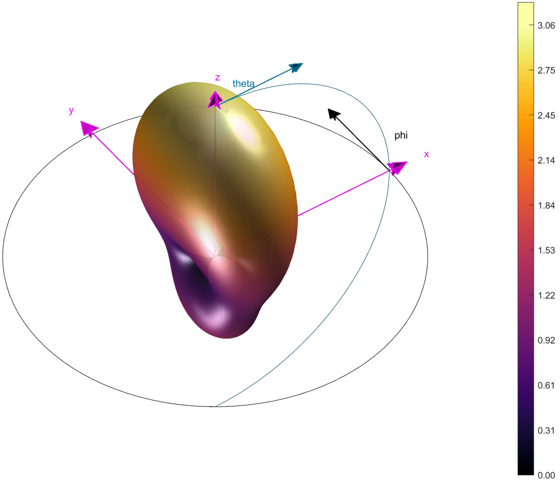


Optimal current for point **B** ($\delta \approx 3.16 \cdot 10^{-4}$).



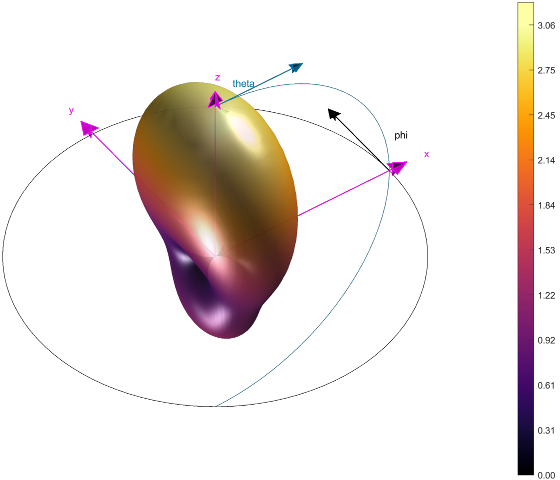
Optimal current for point **C** ($\delta \approx 10^{-3}$).

Comparison of Far Fields $\boldsymbol{F}_0$ and $\boldsymbol{F}$



Desired far field $\boldsymbol{F}_0$.

Comparison of Far Fields $\boldsymbol{F}_0$ and $\boldsymbol{F}$



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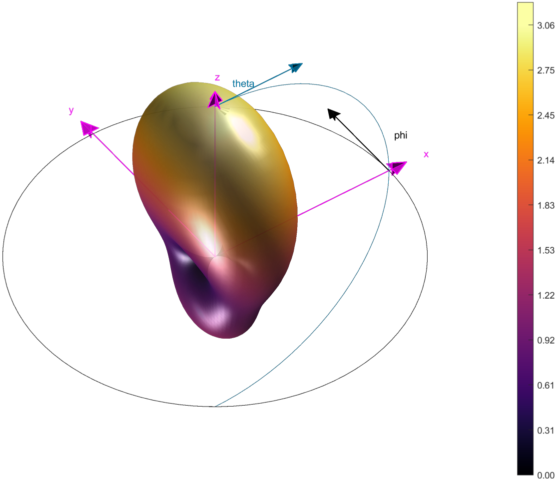

Synthetized far field $\boldsymbol{F}$ points **B** and **C**.

Entire-domain Basis For Compact Far-Field Representation

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- No possibility to further restrict the solution.
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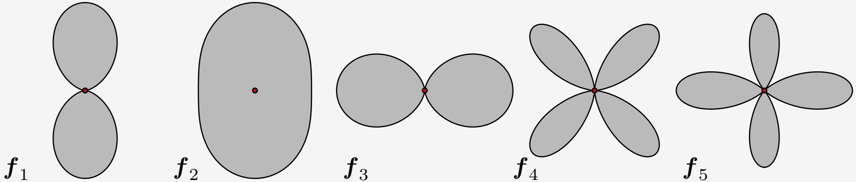
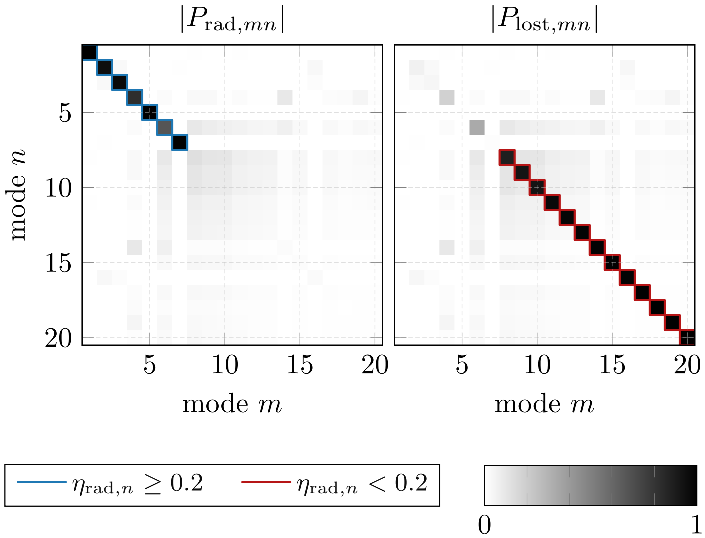
Lossy Characteristic Modes

$$\mathbf{X}\mathbf{I}_n = \lambda_n \left(\mathbf{R} + \mathbf{L}\right) \mathbf{I}_n$$

- Potentially sparse basis of entire-domain functions.
- Relation to Lebedev quadrature and required number of points.
- Different properties than the classical characteristic modes ($\mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}\mathbf{I}_n$).



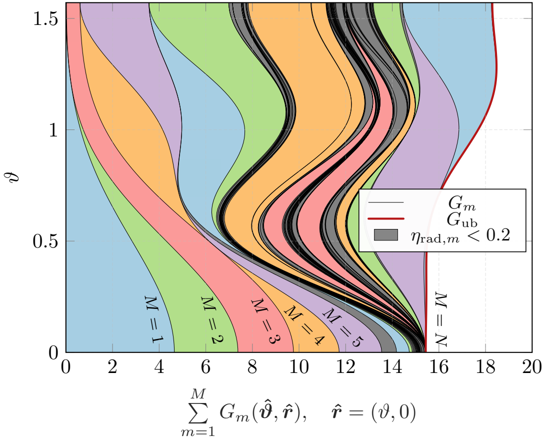


Power terms $P_{\mathrm{rad},mn} = \mathbf{I}_m^{\mathrm{H}} \mathbf{R} \mathbf{I}_n / 2$ (left) and $P_{\mathrm{lost},mn} = \mathbf{I}_m^{\mathrm{H}} \mathbf{L} \mathbf{I}_n / 2$ (right).
Two rectangular plates $\ell \times \ell/2$, separated by $\ell/4$, made of copper, $k\ell = \pi$.

Maximum Gain as Inherent Property of LCMs

It can be shown[9] that LCMs follows:

$$G_{\mathrm{ub}}\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right) = \sum_n G_n\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right)$$



- Curiously enough, the property above was unknown to both Harrington and Garbacz!

[9] M. Capek and L. Jelinek, "Fundamental bound on maximum gain as a sum of lossy characteristic modes and its feasibility,", 2023, eprint arXiv: 2302.06425. [Online]. Available: https://arxiv.org/abs/2302.06425

Additional Insight for a Designer: Modal Parameters

Optimal excitation coefficient:

$$\beta_n = \sqrt{\frac{4\pi}{Z_0 G_{\mathrm{ub}} \left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right)}} F_n^* \left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}}\right)$$

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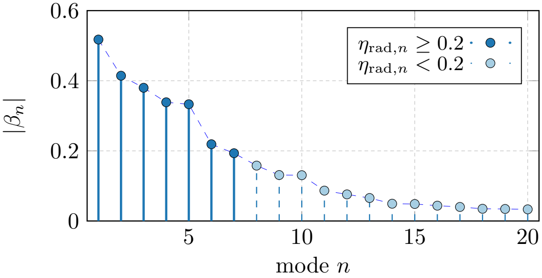

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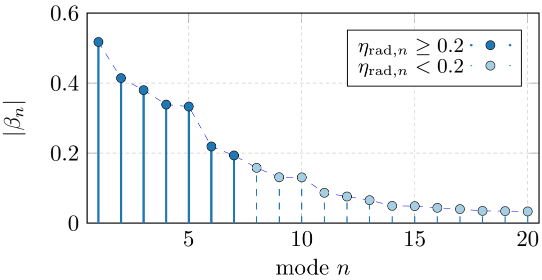

Modal significance:

$$|t_n| = \left| -\frac{1}{1 + \mathrm{j}\lambda_n} \right|$$

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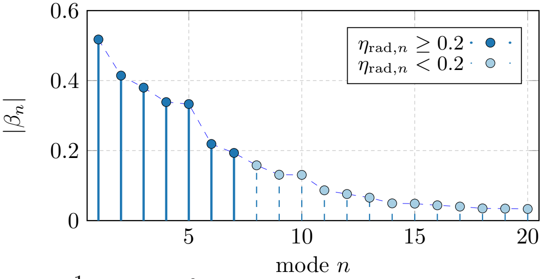

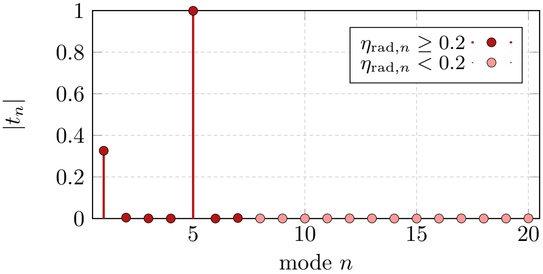

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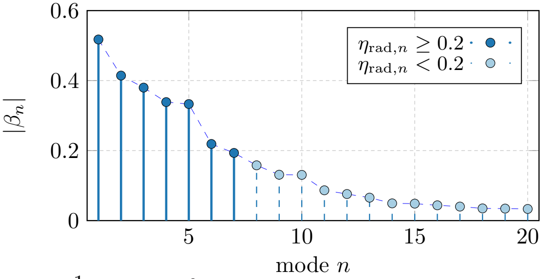

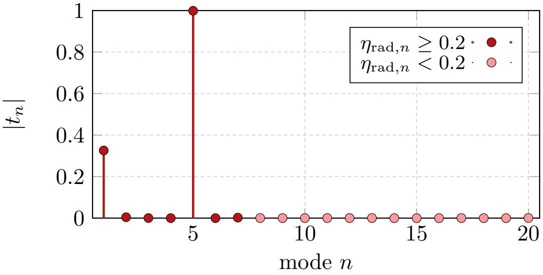

Modal radiation efficiency:

$$\eta_{\mathrm{rad},n} = \frac{P_{\mathrm{rad},nn}}{P_{\mathrm{rad},nn} + P_{\mathrm{lost},nn}} = \mathbf{I}_n^{\mathrm{H}} \mathbf{R} \mathbf{I}_n$$

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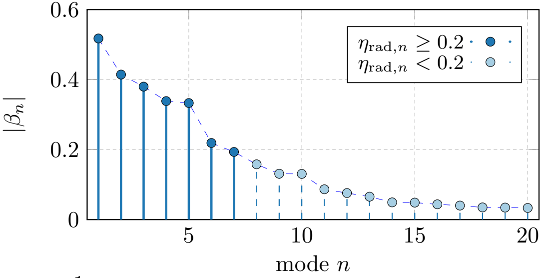

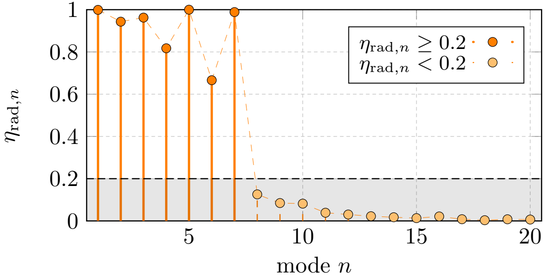

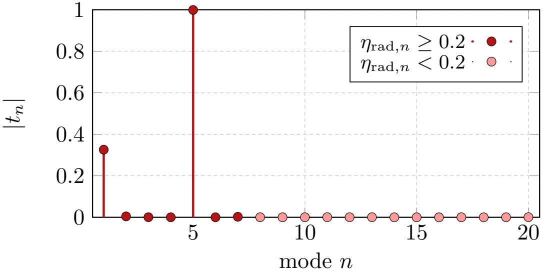

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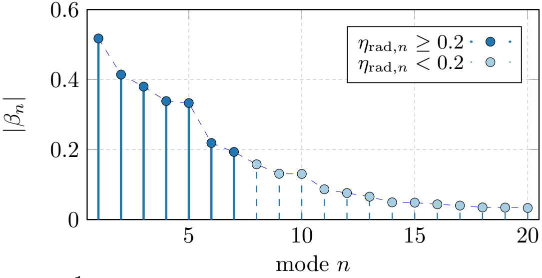
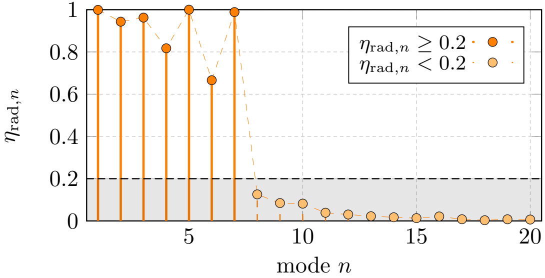
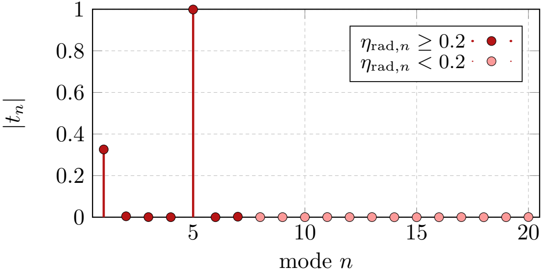


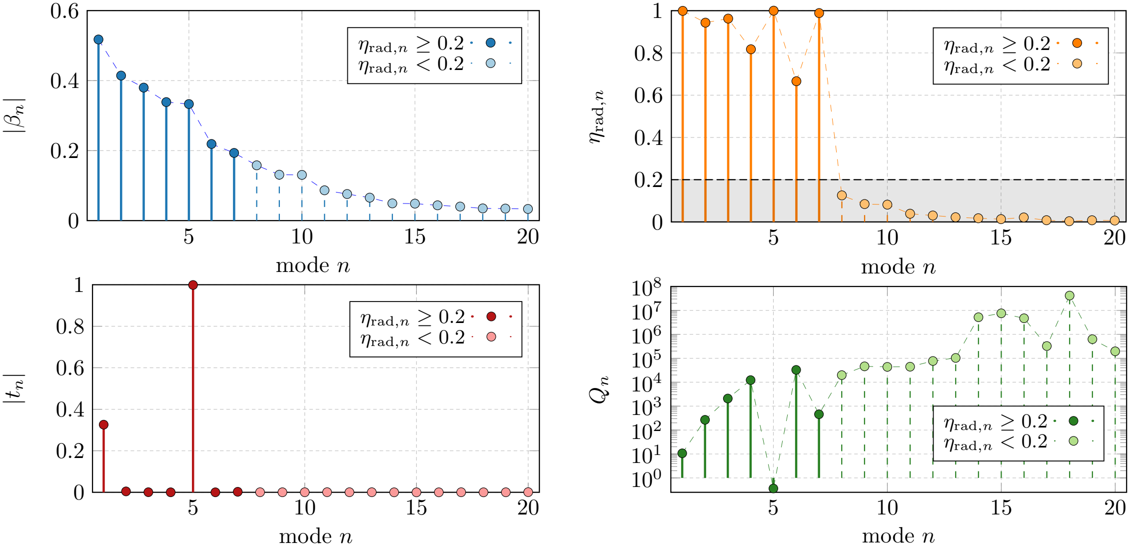




Modal Q-factor:

$$Q_n = \frac{2\omega \max\left\{W_{\mathrm{m},n}, W_{\mathrm{e},n}\right\}}{P_{\mathrm{rad}}}$$

Additional Insight for a Designer: Modal Parameters



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1. Setup geometry, frequency, and material of the design region (Ω, ka, ρ).

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[11] J. Liska, L. Jelinek, and M. Capek, “Fundamental bounds to time-harmonic quadratic metrics in electromagnetism: Overview and implementation,” *arXiv*, 2021. DOI: 10.48550/arXiv.2110.05312

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8. Calculate all associated metrics.

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[11] J. Liska, L. Jelinek, and M. Capek, “Fundamental bounds to time-harmonic quadratic metrics in electromagnetism: Overview and implementation,” *arXiv*, 2021. DOI: 10.48550/arXiv.2110.05312

Problem Solution – The Workflow

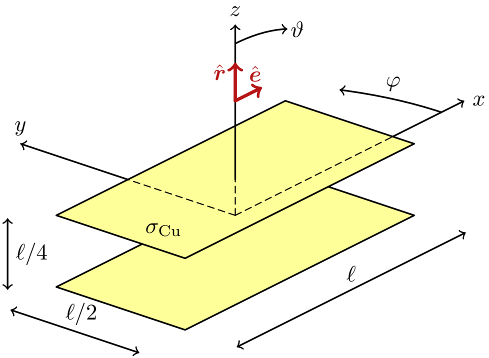
1. Setup geometry, frequency, and material of the design region (Ω, ka, ρ).
2. Solve MoM and evaluate associated operators $\mathbf{R}$, $\mathbf{L}$, $\mathbf{X}$, and $[\mathbf{K}]$ (with AToM[10]).
3. Perform lossy characteristic mode decomposition $\mathbf{X}\mathbf{I}_n = \lambda_n (\mathbf{R} + \mathbf{L}) \mathbf{I}_n$.
4. Analyze modes, determine those being used. For example, $\forall n : \eta_{\mathrm{rad},n}(\mathbf{I}_n) > e$.
5. Project (full) quadratic forms onto reduced basis, *i.e.*, $\mathbf{I} = \sum_k \beta_k \mathbf{I}_k$.
6. Iteratively solve the QCQP for a vector of δ values:
   - Package FunBo from CTU is used[11].
   - Solver QNCQPQuadLin.m is applied (Newton's method over Lagrange multipliers to solve the dual function represented by generalized eigenvalue problem, matrices are factorized with Cholesky and normalized).
7. Reuse Lagrange multipliers found for the next iteration (fast convergence).
8. Calculate all associated metrics.
9. Construct Pareto frontier ($E(\boldsymbol{F}, \boldsymbol{F}_0)$ vs. $\eta_{\mathrm{rad}}(\mathbf{I})$ for each current in Pareto).

[10] "Antenna Toolbox for MATLAB (AToM)," www.antennatoolbox.com, Czech Technical University in Prague. (2019), [Online]. Available: {www.antennatoolbox.com}

[11] J. Liska, L. Jelinek, and M. Capek, "Fundamental bounds to time-harmonic quadratic metrics in electromagnetism: Overview and implementation," *arXiv*, 2021. DOI: 10.48550/arXiv.2110.05312

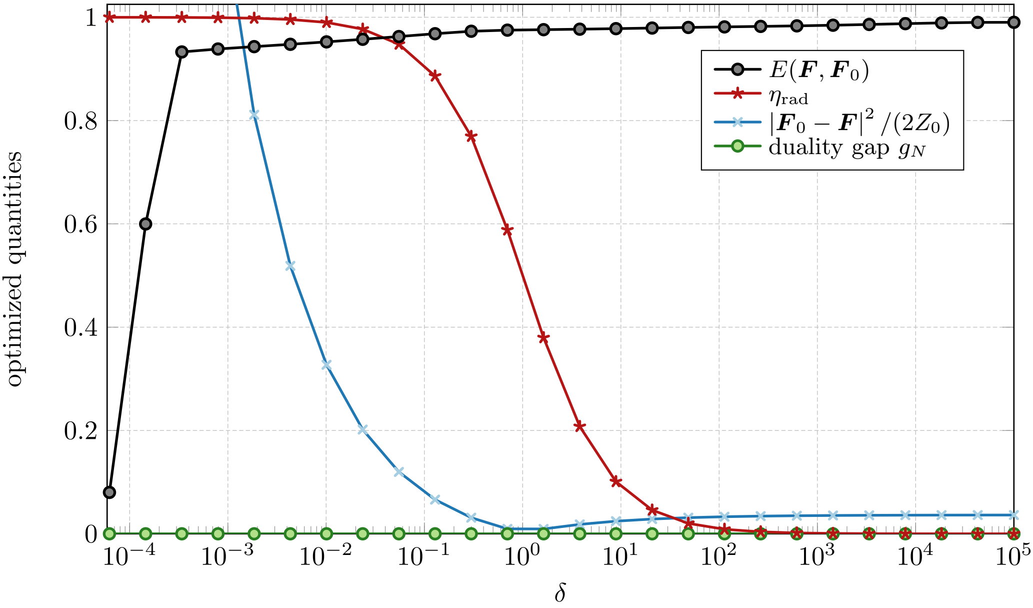
Example #2: Isotropic Far Field

- The same structure and settings as before.
- The desired far field pattern is isotropic in $\hat{\boldsymbol{\varphi}}$ polarization, zero in $\hat{\boldsymbol{\vartheta}}$ polarization.

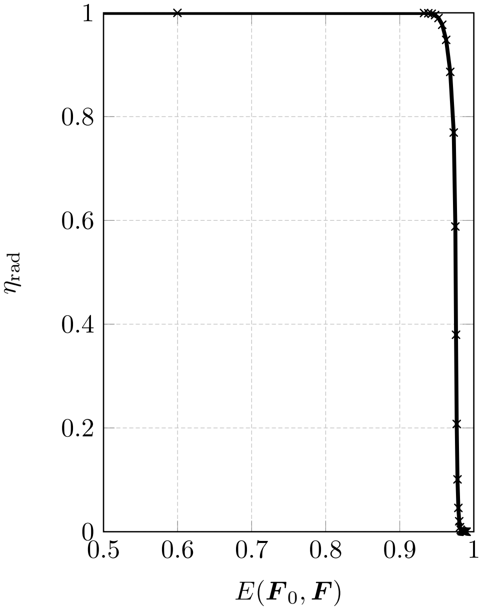


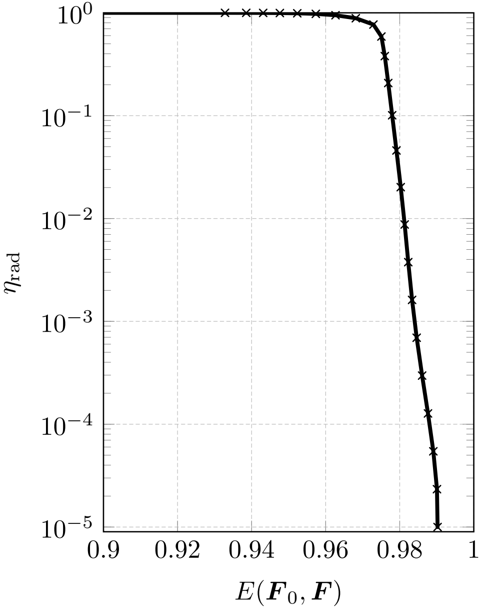
Two parallel plates.

Example #2: Cost Functions

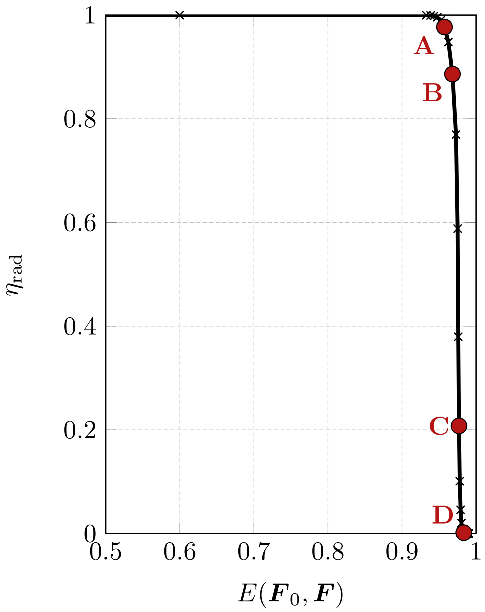


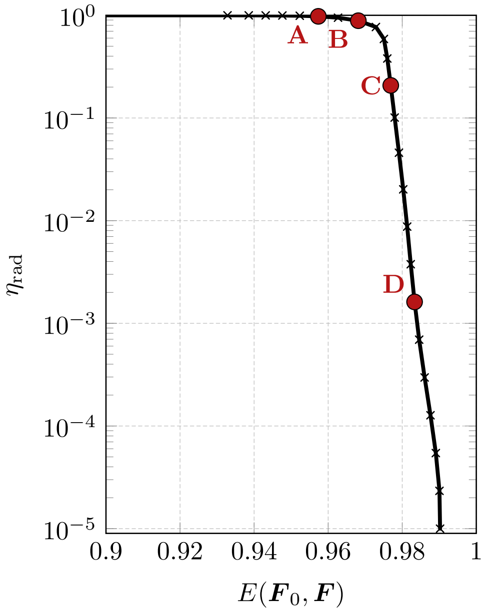
Example #2: Pareto Frontier

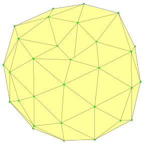


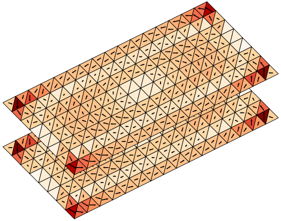


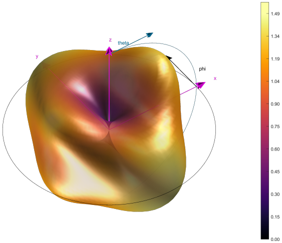
Example #2: Pareto Frontier



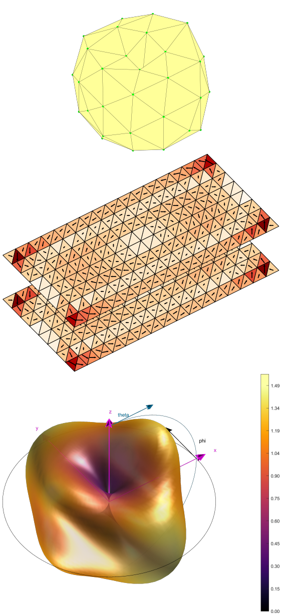


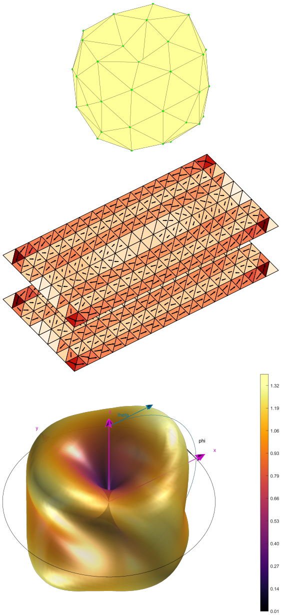






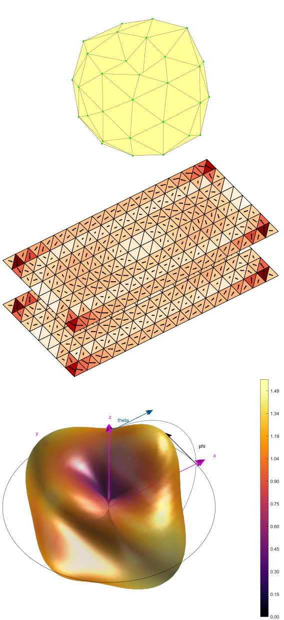
Solution $\mathbf{A}$, $\eta_{\mathrm{rad}} = 0.977$,
$E = 0.957$.





Solution **A**, $\eta_{\mathrm{rad}} = 0.977$, $E = 0.957$.

Solution **B**, $\eta_{\mathrm{rad}} = 0.886$, $E = 0.968$.



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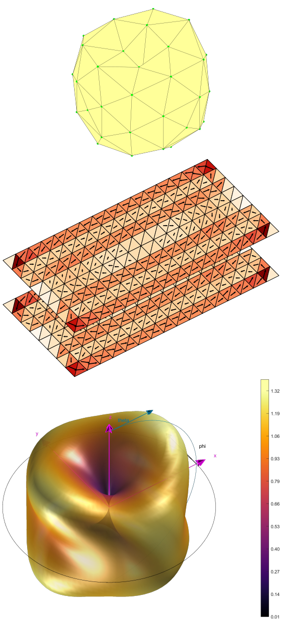
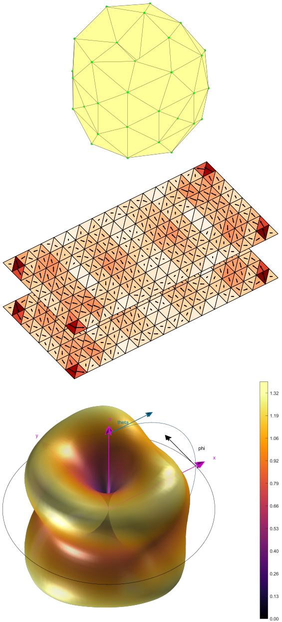
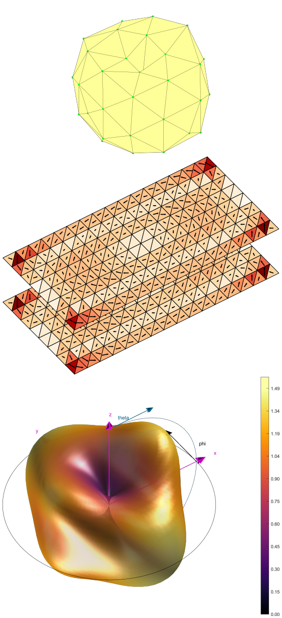


Solution **B**, $\eta_{\mathrm{rad}} = 0.886$, $E = 0.968$.



Solution **C**, $\eta_{\mathrm{rad}} = 0.208$, $E = 0.977$.



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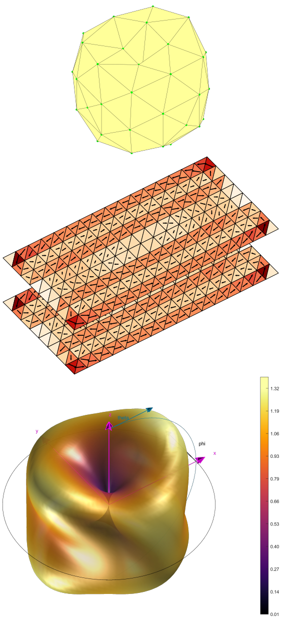
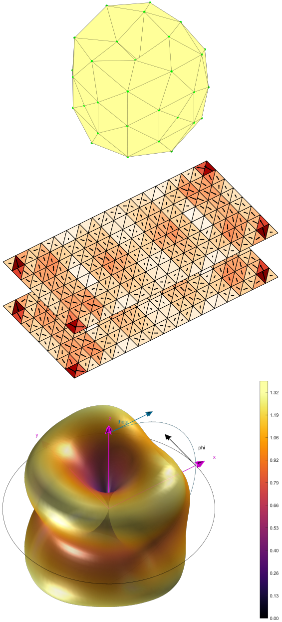


Solution **B**, $\eta_{\text{rad}} = 0.886$, $E = 0.968$.



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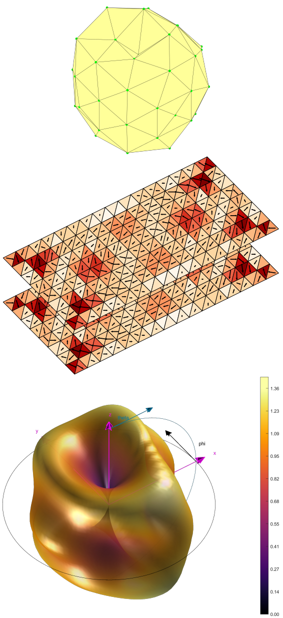
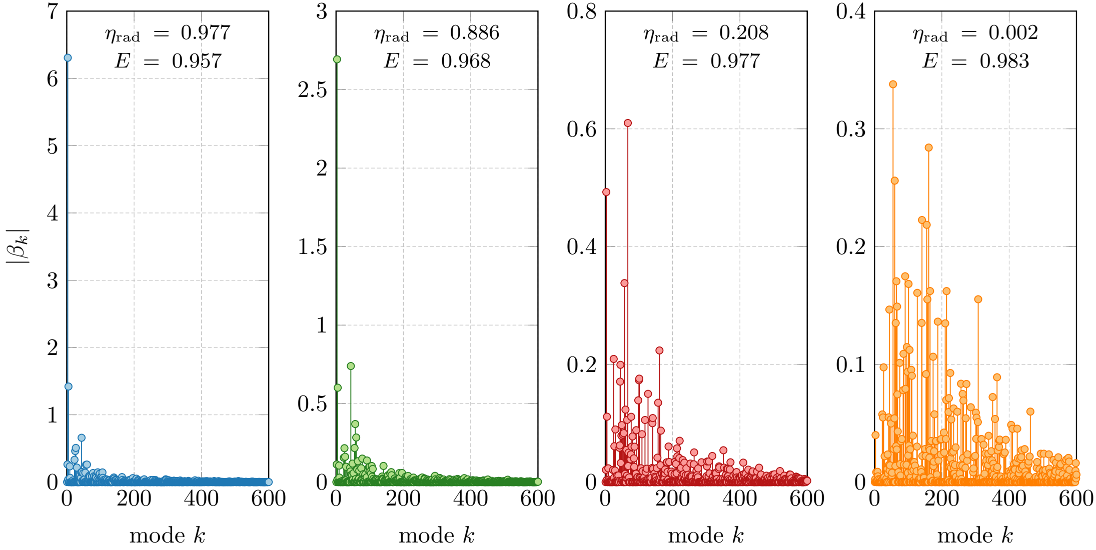


Solution **D**, $\eta_{\text{rad}} = 0.002$, $E = 0.983$.

Example #2: Modal Spectra



Solution – Problem #2

How to approach Problem #2? (Phase of $\boldsymbol{F}_0$ is arbitrary).

$$\begin{aligned} \underset{\mathbf{I}}{\text{minimize}} \quad & \left\| |\boldsymbol{F}_0| - |\boldsymbol{F}(\mathbf{I})| \right\|^2 \\ \text{subject to} \quad & \eta_{\mathrm{rad}}(\mathbf{I}) \leq x \end{aligned}$$

- Suddenly, from easy problem we face an unsolvable one. . .

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Some tricks as before, grouping both constraints together, and the phase is taken as an unknown:

$$\begin{aligned} \underset{\mathbf{I},\mathbf{p}}{\text{minimize}} \quad & -\frac{1}{Z_0} \mathrm{Re}\left(\mathbf{I}^{\mathrm{H}} [\mathbf{K}]^{\mathrm{H}} \mathbf{\Lambda} \mathrm{diag}\left\{\mathbf{F}_0\right\} \mathbf{p}\right) \\ \text{subject to} \quad & \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{L}\mathbf{I} = \delta \\ & \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I} = 1 \end{aligned}$$

where $\mathbf{p} = [p_k]$, $p_k = \exp\left\{\mathrm{j}\phi_n\right\}$.

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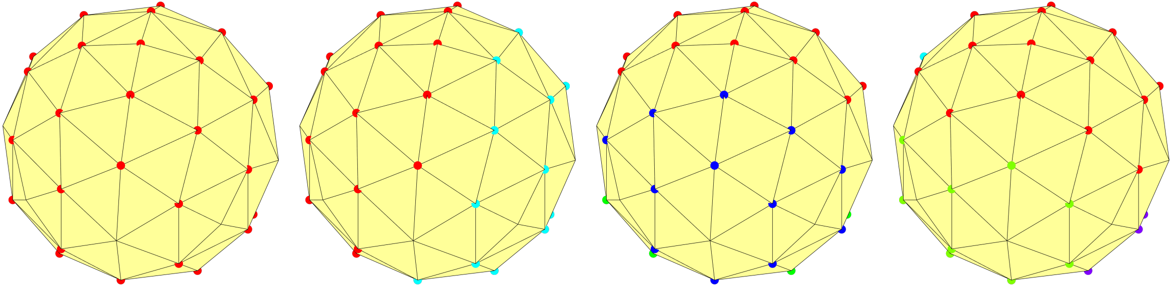

1 cluster in ϑ (8 points skipped).

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3 clusters in ϑ (8 points skipped).

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The Approach Taken...

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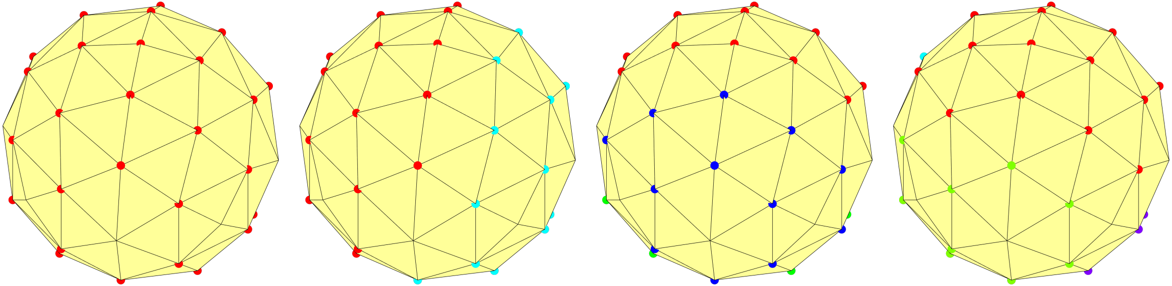
- Performed for ϑ and φ independently. Zeros of $\boldsymbol{F}_0$ can be skipped.



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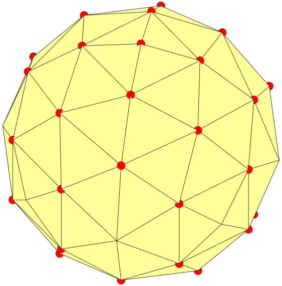

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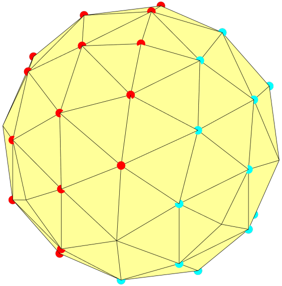
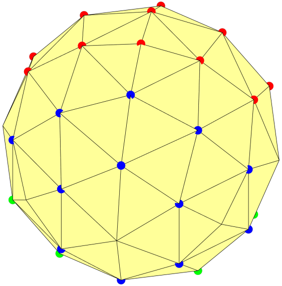
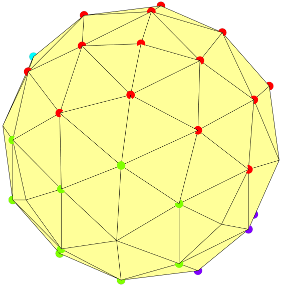


2 clusters in ϑ (8 points skipped).



3 clusters in ϑ (8 points skipped).



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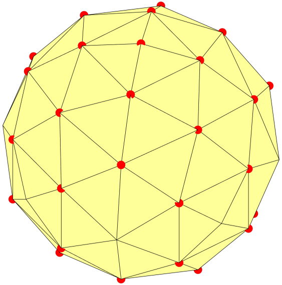
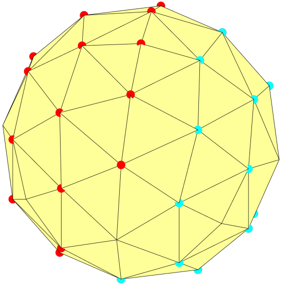
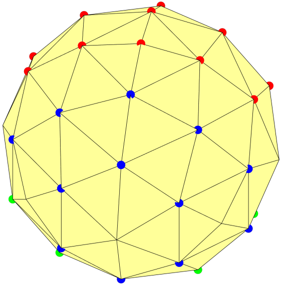
- FMINCON can be replaced by, *e.g.*, manifold optimization[12].



1 cluster in ϑ (8 points skipped).



2 clusters in ϑ (8 points skipped).



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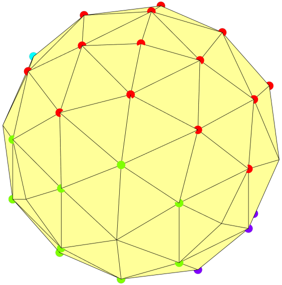


4 clusters in ϑ (8 points skipped).

[12] https://www.manopt.org/

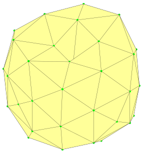
Future Outlook – Test Cases

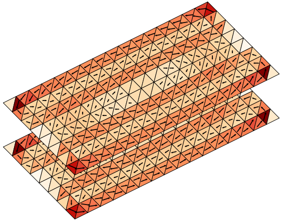
1. What is the cost to replicate far field of one antenna on another (electrically smaller/etc.)?
2. How closely can be, *e.g.*, spherical harmonics radiated by a planar structure?
3. Masked far field (zeros at some places).
4. Close investigation of isotropic radiator (take a spherical shell – no-hair theorem, etc.).
5. What is the cost of pencil beam of different design regions on various parameters?
6. Use projection to port voltages as the only controllable quantities.
7. . . .

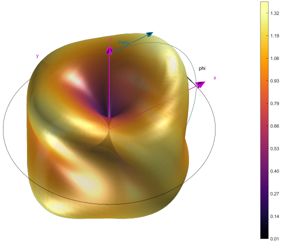
Concluding Remarks

Far field optimiality

- Good problem to think of.
- Mixture of QCQP with other optimization routines.
- Many possible applications...







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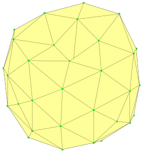
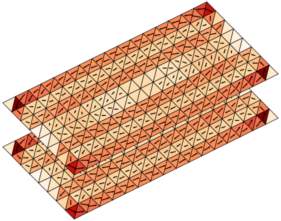
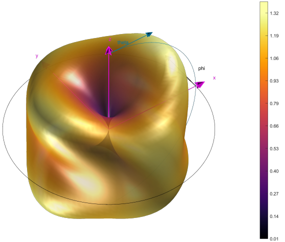
Topics of ongoing research

- To treat Problem #2 (with phase) effectively.
- To try many test cases.
- Investigate cost in Q-factor, excitation constraints.
- Apply port-mode representation (for arrays).







Questions?

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June 23, 2023
version 1.0
The presentation is available at ▶ capek.elmag.org

Acknowledgment: To my wife for letting me go :)